Exercise 30

(a) If $G(x) = 4x^2 - x^3$, find $G'(a)$ and use it to find equations of the tangent lines to the curve $y = 4x^2 - x^3$ at the points $(2, 8)$ and $(3, 9)$.

(b) Illustrate part (a) by graphing the curve and the tangent lines on the same screen.

Solution

Determine the derivative of $G(x)$.

$$\begin{aligned}
G'(x) &= \lim_{h\to 0} \frac{G(x+h) - G(x)}{h} \\
&= \lim_{h\to 0} \frac{[4(x+h)^2 - (x+h)^3] - [4x^2 - x^3]}{h} \\
&= \lim_{h\to 0} \frac{[4(x^2 + 2xh + h^2) - (x^3 + 3x^2h + 3xh^2 + h^3)] - 4x^2 + x^3}{h} \\
&= \lim_{h\to 0} \frac{(4x^2 + 8xh + 4h^2 - x^3 - 3x^2h - 3xh^2 - h^3) - 4x^2 + x^3}{h} \\
&= \lim_{h\to 0} \frac{8xh + 4h^2 - 3x^2h - 3xh^2 - h^3}{h} \\
&= \lim_{h\to 0} (8x + 4h - 3x^2 - 3xh - h^2) \\
&= 8x - 3x^2
\end{aligned}$$

Plug in $x = 2$ to this formula to get $G'(2)$.

$$G'(2) = 8(2) - 3(2)^2 = 4$$

This is the slope of the tangent line to the curve at $x = 2$. Use the point-slope formula and the provided point $(2, 8)$ to get the equation of this line.

$$y - 8 = 4(x - 2)$$

$$y - 8 = 4x - 8$$

$$y = 4x$$

Plug in $x = 3$ to get $G'(3)$.

$$G'(3) = 8(3) - 3(3)^2 = -3$$

This is the slope of the tangent line to the curve at $x = 3$. Use the point-slope formula and the provided point $(3, 9)$ to get the equation of this line.

$$y - 9 = -3(x - 3)$$

$$y - 9 = -3x + 9$$

$$y = -3x + 18$$

Below is a graph of the curve along with the tangent lines at $x = 2$ and $x = 3$.